JOURNAL OF APPROXIMATION THEORY **62**, 381–382 (1990)

Uniqueness for Monotone Reciprocals

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Communicated by D. S. Lubinsky

Received October 13, 1988; revised February 7, 1989

Consider uniform approximation on a compact subset X of an interval by

$$R = \{1/p\colon 1/p \text{ monotone increasing on } X,\ p > 0 \text{ on } X,\ p \in P\},$$

where P is a linear subspace of $C(X)$. This is a problem of monotone approximation [2, p. 65ff]. In [3] was first given

DEFINITION. A subset $\mathscr{G}$ of $C(X)$ is said to have the *betweeness property* if for any two elements G_0 and G_1, there is a λ-set $\{H_\lambda\}$ of elements of $\mathscr{G}$ such that $H_0 = G_0$, $H_1 = G_1$, and for all $x \in X$, $H_\lambda(x)$ is either a constant or a strictly monotone function of λ, $0 \leqslant \lambda \leqslant 1$.

Linear spaces have the betweeness property [3].

LEMMA. *Let $\mathscr{G}$ have betweeness and uniqueness of best approximations, then so do the elements of $\mathscr{G}$ which are > 0 on X.*

Proof. Consider the definition of zero-sign compatibility in [3, p. 155]: by betweeness F can be chosen > 0 on X. This can be achieved due to [3, Lemma 1], the strict monotoneness of λ-sets at non-constant values, and the fact that positive elements of $\mathscr{G}$ are bounded away from zero.

THEOREM. *Let best approximation by monotone decreasing elements of P be unique, then best approximation by R is unique.*

Proof. Suppose not. Arguments before Theorem 4 of the author's paper [4] establish that R has the betweeness property. Hence R fails to have zero-sign compatibility [3, p. 155]; that is, there are $1/p$, $1/q$ distinct in R and a continuous function s taking absolute value 1 on a closed subset Z of the zeros of $1/p - 1/q$ such that no $1/r$ exists in R with

$$\operatorname{sgn}[1/p(x) - 1/r(x)] = s(x), \qquad x \in Z.$$

0021-9045/90 $3.00

As $1/p - 1/q = (q-p)/(pq)$, Z is a closed subset of the zeros of $p-q$ and no monotone decreasing r exists in P, $r>0$ on X, such that

$$\operatorname{sgn}[p(x)-r(x)] = -s(x), \qquad x \in Z,$$

but p, q are monotone decreasing, hence the positive monotone decreasing elements of P do not have zero-sign compatibility, hence uniqueness fails, contradiction.

Remark. If we restrict ourselves to *positive* monotone decreasing elements of P, a set which has betweeness by the cited arguments of [4], it is seen that non-uniqueness here implies non-uniqueness in R by reversing the equations.

Classical monotone linear uniqueness results may be couched in terms of monotone *increasing* elements of P: to apply these we might reverse the interval. For example the interval $[-1, 2]$ would be replaced by $[-2, 1]$: monotone decreasing functions $g(x)$ would be replaced by monotone increasing functions $g(-x)$.

The "reciprocals" given by the author in [4], are somewhat richer, being composed of elements >0, $=0$, and <0. As none of the three classes can touch, identical arguments for elements >0 and similar arguments for <0 give a similar theory.

References

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